

VARIATION OF BERGMAN KERNELS OF PSEUDOCONVEX DOMAINS

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ABSTRACT. We shall give a variational formula of the full Bergman kernels associated to a family of smoothly bounded strongly pseudoconvex domains. An equivalent criterion for the triviality of holomorphic motions of planar domains in terms of the Bergman kernel is given as an application.

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1. INTRODUCTION

This paper is an attempt to generalize Berndtsson's result on the plurisubharmonicity property of the Bergman kernel. Let ϕ be a plurisubharmonic function on a pseudoconvex domain D in $\mathbb{C}_t^m \times \mathbb{C}_{\zeta}^n$. Denote by $K^t(\zeta, \bar{\eta})$ the associated weighted (full) Bergman kernel on fibre D_t . Berndtsson (see Theorem 1.1 in [1]) proved that $\log K^t(\zeta, \bar{\zeta})$ is plurisubharmonic on D (see also [20] and [21] for early results in this direction). The most important ingredient in his proof is a curvature property (denote by **C**) on product domain (see [2],[3],[5] and [6] for applications). Our start point is to translate **C** to plurisubharmonicity property of the Bergman projection. Since Bergman projection has extremal property, the approximation technique in [1] (i.e., from product case to general case) still applies. Thus we get plurisubharmonicity property of the Bergman projection for general D and ϕ . Then plurisubharmonicity property of the Bergman kernel can be seen as a special case (i.e., one point evaluation). By virtue of this observation, it is more natural to study variation of full Bergman kernels $K^t(\zeta, \bar{\eta})$ (not only Bergman kernels on the diagonal), which is the

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theme of this paper. In order to get the second order variation formula for the Bergman kernel, it is necessary to find a $(1, 0)$ - vector field such that the associated representative of the Kodaira-Spencer class is primitive with respect to some complete Kähler metric. We shall show that how to use quasi-Kähler-Einstein metric to construct such vector field.

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2. BASIC NOTIONS AND RESULTS

Denote by \mathbb{B} the unit ball in \mathbb{C}^m . Let $\{D_t\}_{t \in \mathbb{B}}$ be a family of smoothly bounded strongly pseudoconvex domains in \mathbb{C}^n . Assume that

$$D := \{(t, \zeta) \in \mathbb{C}^{m+n} : \zeta \in D_t, t \in \mathbb{B}\}$$

possesses a smooth defining function ρ such that $\rho^t := \rho|_{D_t} \in C^\infty(\overline{D_t})$ is a strictly plurisubharmonic defining function of D_t for every t in D .

Let ϕ be a smooth real function such that ϕ is smooth on a neighborhood of $\bar{D} \cap (\mathbb{B} \times \mathbb{C}^n)$. Put

$$\mathcal{H}_t := \{f \in H^0(D_t, \wedge^n T^* D_t) : i^{n^2} \int_{D_t} f \wedge \bar{f} e^{-\phi} < \infty\}.$$

Denote by $K^t(\zeta, \bar{\eta}) d\zeta \wedge \overline{d\eta}$ the Bergman kernel of \mathcal{H}_t , where $d\zeta$ is short for $d\zeta^1 \wedge \cdots \wedge d\zeta^n$. Using Hamilton's theory on families of non-coercive boundary value problems (see [11], [12] or [10]), one may prove that:

Lemma 2.1. *For every fixed $\eta \in D_{t_0}$, $t_0 \in \mathbb{B}$, $K^t(\zeta, \bar{\eta})$ is smooth (as a function of $t_j, \bar{t}_k, \zeta_\alpha$) on a neighborhood of $\bar{D} \cap (U_{t_0} \times \mathbb{C}^n)$, where U_{t_0} is a neighborhood of t_0 .*

Put $\cdot \bar{\eta} = K^t(\mu, \bar{\eta}) d\mu$. The above lemma implies that

$$\cdot \bar{\eta}_{\bar{j}} := \frac{\partial}{\partial t^j} K^t(\mu, \bar{\eta}) d\mu \in \mathcal{H}_t,$$

for every fixed $\eta \in D_t$. By reproducing property of the Bergman kernels, we have

$$(2.1) \quad K_{\bar{j}}(\zeta, \bar{\eta}) = \langle \cdot \bar{\eta}_{\bar{j}}, \cdot \bar{\zeta} \rangle, \quad K_j(\zeta, \bar{\eta}) = \langle \cdot \bar{\eta}, \cdot \bar{\zeta}_j \rangle.$$

There is another way to compute $K_j(\zeta, \bar{\eta})$ (without using reproducing property), i.e., to compute

$$\frac{\partial}{\partial t^j} \int_{D_t} \{ \cdot \bar{\eta}, \cdot \bar{\eta} \},$$

where $\{ \cdot, \cdot \} := \cdot \wedge \bar{\cdot} e^\phi$ is the canonical sesquilinear pairing. Denote by P the canonical projection from $\mathbb{B} \times \mathbb{C}^n$ to \mathbb{B} . Put

$$\mathcal{V}_j = \{V_j \in T(\rho) : P_* V_j = \partial / \partial t^j\},$$

where $T(\rho)$ is the space of all $(1, 0)$ -vector field V such that V is smooth on neighborhood of $\bar{D} \cap (\mathbb{B} \times \mathbb{C}^n)$ and $V(\rho)$ vanishes on $\partial D \cap (\mathbb{B} \times \mathbb{C}^n)$. Denote by i_t the inclusion mapping $D_t \hookrightarrow D$. Since vector fields in $T(\rho)$ are tangent to the boundary, we can prove that

$$(2.2) \quad \frac{\partial}{\partial t^j} \int_{D_t} \cdot = \int_{D_t} L_{V_j} \cdot = \int_{D_t} L_{V_j}^t \cdot, \quad \forall V_j \in \mathcal{V}_j.$$

where L_{V_j} is the usual Lie-derivative and $L_{V_j}^t := i_t^* L_{V_j}$. Thus we have

$$K_j(\zeta, \bar{\eta}) = i^{n^2} \int_{D_t} L_{V_j} \{ \cdot \bar{\eta}, \cdot \bar{\zeta} \}.$$

By Cartan's formula

$$L_{V_j} = d\delta_{V_j} + \delta_{V_j}d, \quad \delta_{V_j} \text{ means contraction of a form with } V_j,$$

thus we have

$$K_j(\zeta, \bar{\eta}) = i^{n^2} \int_{\partial D_t} \delta_{V_j} \{\cdot \bar{\eta}, \cdot \bar{\zeta}\} + i^{n^2} \int_{D_t} \frac{\partial}{\partial t^j} \{\cdot \bar{\eta}, \cdot \bar{\zeta}\}.$$

By (2.1), the second term of the right hand side of the above equality is

$$2K_j(\zeta, \bar{\eta}) - i^{n^2} \int_{D_t} \phi_j \{\cdot \bar{\eta}, \cdot \bar{\zeta}\},$$

thus we get Hadamard's first variational formula

$$(2.3) \quad K_j(\zeta, \bar{\eta}) = i^{n^2} \int_{D_t} \phi_j \{\cdot \bar{\eta}, \cdot \bar{\zeta}\} - i^{n^2} \int_{\partial D_t} \delta_{V_j} \{\cdot \bar{\eta}, \cdot \bar{\zeta}\},$$

which is due to Komatsu [16] (in case $\phi \equiv 0$).

Notice that

$$\mathcal{L}_{V_j}^t \{\cdot, \cdot\} = \{L_j \cdot, \cdot\} + \{\cdot, L_{\bar{j}} \cdot\},$$

where L_j (resp. $L_{\bar{j}}$) are short for $e^\phi L_{V_j}^t(e^{-\phi})$ (resp. $L_{V_{\bar{j}}}^t$). Since $L_{\bar{j}} \cdot \bar{\eta} = \cdot \bar{\eta}_{\bar{j}}$, (2.1) implies that $L_j \cdot \bar{\eta} \perp \mathcal{H}_t$. Thus we have

$$0 = \langle \langle L_j \cdot \bar{\eta}, \cdot \bar{\zeta} \rangle \rangle_{\bar{k}} = i^{n^2} \int_{D_t} \{L_j \cdot \bar{\eta}, L_k \cdot \bar{\zeta}\} + \{L_{\bar{k}} L_j \cdot \bar{\eta}, \cdot \bar{\zeta}\},$$

which implies that

$$(2.4) \quad K_{j\bar{k}}(\zeta, \bar{\eta}) = \langle \langle \cdot \bar{\eta}_{\bar{k}}, \cdot \bar{\zeta}_{\bar{j}} \rangle \rangle + \langle \langle [L_j, L_{\bar{k}}] \cdot \bar{\eta}, \cdot \bar{\zeta} \rangle \rangle - i^{n^2} \int_{D_t} \{L_j \cdot \bar{\eta}, L_k \cdot \bar{\zeta}\}.$$

Denote by ${}_j \bar{\eta}^{n,0}$ (resp. ${}_j \bar{\eta}^{n-1,1}$) the $(n,0)$ (resp. $(n-1,1)$) -part of $L_j \cdot \bar{\eta}$. By Cartan's formula, we have

$${}_j \bar{\eta}^{n,0} = i_t^* e^\phi (\partial \delta_{V_j} + \delta_{V_j} \partial) (e^{-\phi} \cdot \bar{\eta}), \quad {}_j \bar{\eta}^{n-1,1} = \delta_{\bar{\partial}^t V_j} \cdot \bar{\eta}.$$

In deformation theory (see [15]), $\bar{\partial}^t V_j$ represent Kodaira-Spencer classes.

Now the third term of the right hand side of (2.4) can be written as a Hermitian form

$$I_3 = -\langle \langle {}_j \bar{\eta}^{n,0}, {}_k \bar{\zeta}^{n,0} \rangle \rangle - i^{n^2} \int_{D_t} \{{}_j \bar{\eta}^{n-1,1}, {}_k \bar{\zeta}^{n-1,1}\}.$$

Since ${}_j \bar{\eta}^{n,0} \perp \mathcal{H}_t$, ${}_j \bar{\eta}^{n,0}$ is the L^2 -minimal solution of

$$\bar{\partial}^t(\cdot) = \bar{\partial}^t({}_j \bar{\eta}^{n,0}).$$

Notice that

$$(2.5) \quad -\bar{\partial}^t({}_j \bar{\eta}^{n,0}) = \partial_\phi^t({}_j \bar{\eta}^{n-1,1}) + (\bar{\partial}^t \phi)_{V_j} \wedge \cdot \bar{\eta},$$

where $\partial_\phi^t := e^\phi \partial^t(e^{-\phi})$ and $(\bar{\partial}^t \phi)_{V_j} = \sum (V_j \phi_{\bar{\alpha}}) d\bar{\mu}^\alpha$. We shall use L^2 -estimates to decode the positivity of I_3 . In this way, we need that ${}_j \bar{\eta}^{n-1,1}$ is **primitive with respect to some complete Kähler metric** on D_t .

Primitivity is natural in the sense that

$$-i^{n^2} \int_{D_t} \{\cdot, \cdot\} = \langle \langle \cdot, \cdot \rangle \rangle$$

for primitive $(n-1, 1)$ -form. But if the metric is not complete, by Hörmander's density lemma (see [13] and [14]), we have to estimate

$$\langle \langle \partial_\phi^t({}_j\bar{\eta}^{n-1,1}), u \rangle \rangle, \quad \forall u \in \text{Dom}(\bar{\partial}^t)^* \cap \ker \bar{\partial}^t \cap C^\infty(\bar{D}_t).$$

In order to use

$$(2.6) \quad \langle \langle \partial_\phi^t({}_j\bar{\eta}^{n-1,1}), u \rangle \rangle = \langle \langle {}_j\bar{\eta}^{n-1,1}, (\partial_\phi^t)^* u \rangle \rangle,$$

we have to assume that $u \in \text{Dom}(\partial_\phi^t)^*$, which may not be true for $n \geq 2$.

Notice that ${}_j\bar{\eta}^{n-1,1}$ is primitive with respect to Kähler form ω^t if

$$(2.7) \quad \bar{\partial}^t(\delta_{V_j}\omega^t) = 0.$$

Since ω^t is a Kähler metric. (2.7) is equivalent to the primitivity of the Kodaira-Spencer class. Now it suffices to find $V_j \in \mathcal{V}_j$ such that $\delta_{V_j}\omega^t$ is $\bar{\partial}^t$ -closed for some complete Kähler metric ω^t . In general, if there exists ψ such that $i\partial^t\bar{\partial}^t\psi = \omega^t$, then the unique solution $V := V_j^\psi$ of

$$\delta_V\omega^t = -i\bar{\partial}^t\psi_j, \quad P_*V = \partial/\partial t^j$$

of course satisfies (2.7). Now we only need to **find ψ such that ω^t is complete and V_j^ψ is tangent to the boundary** (i.e., $V_j^\psi \in \mathcal{V}_j$).

By definition, $\{D_t\}$ is a smooth family of pseudoconvex domains with strictly plurisubharmonic definition functions $\rho|_{D_t}$. Thus

$$\omega^t := i\partial^t\bar{\partial}^t(-\log -\rho).$$

is a complete Kähler metric on D_t . We call ω^t quasi-Kähler-Einstein metric on D_t . In fact, by Cheng-Yau's result [7], one may choose defining function ρ_{CY} such that $i\partial^t\bar{\partial}^t(-\log -\rho_{CY})$ is a complete-Kähler-Einstein metric on D_t . To our surprise, every quasi-Kähler-Einstein metric fits our needs:

Key Lemma. *Every $V_j^{-\log -\rho}$ is tangent to the boundary (i.e., $V_j^{-\log -\rho} \in \mathcal{V}_j$).*

Let's go back to (2.4). Using integration by parts, we shall prove that the second term of the right hand side of (2.4) contains a boundary term with density

$$(2.8) \quad b_{j\bar{k}}(\rho) := \frac{\langle V_j^{-\log -\rho}, V_k^{-\log -\rho} \rangle_{i\partial\bar{\partial}\rho}}{|\partial\rho|}.$$

Denote by ${}_j\bar{\eta}$ the \square' -harmonic part of ${}_j\bar{\eta}^{n-1,1}$, where \square' denotes the ∂_ϕ^t -Laplace. Using our Key Lemma and L^2 -estimates on complete Kähler manifolds [8], we can prove that:

Theorem 2.2. *If $\phi \equiv 0$ on D then*

$$(2.9) \quad K_{j\bar{k}}(\zeta, \bar{\eta}) = \langle \langle \bar{\eta}_{\bar{k}}, \bar{\zeta}_{\bar{j}} \rangle \rangle + \int_{\partial D_t} b_{j\bar{k}}(\rho) \langle \bar{\eta}, \bar{\zeta} \rangle d\sigma + \langle \langle {}_j\bar{\eta}, {}_k\bar{\zeta} \rangle \rangle,$$

where $d\sigma$ is the surface measure on ∂D_t and $\langle \bar{\eta}, \bar{\zeta} \rangle := K^t(\mu, \bar{\eta}) \overline{K^t(\mu, \bar{\zeta})} e^{-\phi^t(\mu)}$.

As a direct corollary, we have:

Corollary 2.3. *If $\phi \equiv 0$ and D is pseudoconvex then $\{\mathcal{H}_t\}$ is Nakano semi-positive.*

Before we prove it, it is necessary to give a definition of the notion of Nakano positivity for a family of Hilbert spaces with infinite rank.

In finite dimensional (compact fibres) case, one may use variation of Bergman kernels to give an equivalent definition of Nakano positivity for direct image bundle $\{\mathcal{H}_t\}$. Put

$$K_{j\bar{k}p\bar{q}} = K_{j\bar{k}}(\bar{\eta}_q, \bar{\eta}_p) - \langle \cdot, \bar{\eta}_{p,\bar{k}}, \cdot \bar{\eta}_{q,\bar{j}} \rangle,$$

where η_p, η_q are points in D_t , $1 \leq p, q \leq r$, $r \in \mathbb{Z}_+$. We shall prove that (see Lemma 4.5) Nakano semi-positivity of $\{\mathcal{H}_t\}$ is equivalent to

$$(2.10) \quad \sum c^{jp} \bar{c}^{kq} K_{j\bar{k}p\bar{q}} \geq 0,$$

where $r = \dim_{\mathbb{C}} \mathcal{H}_t$. In case $\dim_{\mathbb{C}} \mathcal{H}_t = \infty$, we say that $\{\mathcal{H}_t\}$ is Nakano semi-positive if the corresponding Bergman kernels satisfies (2.10) for all $r \in \mathbb{Z}_+$.

By a similar argument as in the proof of Theorem 2.2, we can also get a variational formula for weighted Bergman kernel. But it turns out to be an inequality instead of an equality. In case the weight function is plurisubharmonic, we shall prove that the corresponding direct image sheaf is Nakano semi-positive which can be seen as a generalization of Berndtsson's result (Theorem 1.1 in [2]) to non-product case (see also [27], [24], [23] and [18] for other results in this direction).

Assume that our weight function ϕ is smooth up to the boundary of D . Assume further that ϕ is strictly plurisubharmonic along the fibres. Thus geodesic curvature of $\{\phi^t\}$

$$(2.11) \quad c_{j\bar{k}}(\phi) := \phi_{j\bar{k}} - \sum \phi_{j\bar{\alpha}} \phi^{\bar{\alpha}\beta} \phi_{\bar{k}\beta}.$$

is well defined. By Berndtsson's formula (see Lemma 4.1 in [4]),

$$\delta_{V_j^\phi}(\partial\bar{\partial}\phi) = \sum c_{j\bar{k}}(\phi) d\bar{t}^k.$$

We show prove that:

Theorem 2.4. *If D is pseudoconvex and ϕ is plurisubharmonic on D then $\{\mathcal{H}_t\}$ is Nakano semi-positive.*

Berndtsson has informed the author that it would be also possible to define the notion of curvature $\Theta_{j\bar{k}}$ (as a densely defined closed operator) for $\{\mathcal{H}_t\}$. Then the proof of the above theorem implies that $\Theta_{j\bar{k}} - c_{j\bar{k}}(\phi)$ is Nakano semi-positive if D is pseudoconvex.

Recall that our start point is to translate the curvature property to plurisubharmonicity property of the Bergman projection. Let f be a smooth function with compact support in D such that f is a holomorphic function of t . Put

$$(2.12) \quad K_f(t) = \int_{D_t \times D_t} K^t(z, \bar{w}) f(t, z) \overline{f(t, w)}.$$

Thus $K_f(t)$ is the square norm of the Bergman projection of $\bar{f}e^{\phi^t}$. Let u^t be the L^2 -minimal solution of $\bar{\partial}^t(\cdot) = \bar{\partial}^t(\bar{f}e^{\phi^t})$. Then $K_f(t) = \|\bar{f}e^{\phi^t}\|^2 - \|u^t\|^2$. In compact case, Berndtsson and Păun showed that plurisubharmonicity of K_f is equivalent to Griffiths positivity of the direct image bundle (see Proposition 3.4 in [6]). We shall prove that:

Theorem 2.5. *The function $\log K_f(t)$ is plurisubharmonic on \mathbb{B} .*

Remark. *Using approximation technique in [1], we shall prove that the above theorem is true for general pseudoconvex domain D and plurisubharmonic function ϕ (i.e., no restriction on regularity of D and ϕ).*

We want to point out that Theorem 1.3 in [28] is not true in general (i.e., the support of f is necessary to be relatively compact in D). In fact, take $\phi = 0$, if the above theorem is true for $f = 1$ then $\log |D_t|$ is plurisubharmonic. However, Berndtsson showed that if D_t is invariant under rotations $\cdot \mapsto e^{i\theta} \cdot$ then $-\log |D_t|$ is plurisubharmonic (see Theorem 1.2 in [6]).

Let's go back to (2.4) again. In one-dimensional case, (2.6) is always true. What's more, every form is primitive. Thus every vector field in \mathcal{V}_j can be used to compute the variation. In compact case (i.e., deformation of compact Riemann surfaces), Berndtsson (see [4]) showed that even if the curvature of the 0-th direct image of the relative canonical line bundle vanishes identically, the Kodaira-Spencer class still happened to be non zero (i.e., the deformation is not trivial). Inspired by Berndtsson's idea, we shall use curvature of $\{\mathcal{H}_t\}$ to study triviality of holomorphic motions.

A homeomorphism $F : (t, z) \mapsto (t, f(t, z))$ from $\mathbb{B} \times D_0$ to D is called a holomorphic motion (see [22]) of D_0 (with graph D) if $f(0, \cdot)$ is the identity mapping and $f(\cdot, z)$ is holomorphic for every fixed $z \in D_0$. F is said to be a trivial motion of D_0 if there exists a bi-holomorphic mapping G from $\mathbb{B} \times D_0$ to the graph of F such that $G(\{t\} \times D_0) = F(\{t\} \times D_0)$ for every $t \in \mathbb{B}$ (i.e., there exists a fibre-preserving bi-holomorphic mapping from $\mathbb{B} \times D_0$ to D).

Consider the classical (i.e., $\phi \equiv 0$) Bergman space \mathcal{H}_t of the fibre $D_t := F(\{t\} \times D_0)$. If D_0 is a planar domain then the complex structure on each fibre can be represented by $J = f_{\bar{z}}/f_z$. We shall use variation of the Bergman kernels K^t (or the curvature $\Theta_{j\bar{k}}$ of $\{\mathcal{H}_t\}$) to decode triviality of F .

Theorem 2.6. *Let D_0 be a smoothly bounded planar domain. Let F be a holomorphic motion of D_0 . If F is smooth up to the boundary then the followings are equivalent:*

- (i) F is trivial.
- (ii) $\Theta_{j\bar{k}} \equiv 0$, (i.e., $\sum c^{jp} \overline{c^{kq}} K_{j\bar{k}p\bar{q}} \equiv 0$).
- (iii) For every (t, η) in D and every j ,

$$(2.13) \quad \int_{D_t} K^t(\zeta, \bar{\eta}) \left(\frac{(f_z)^2 J_j}{|f_z|^2 (1 - |J|^2)} \right) (t, \zeta) \, id\zeta \wedge d\bar{\zeta} = 0.$$

As a direct corollary, we have:

Corollary 2.7. *Let $F : (t, z) \mapsto (t, z + a(t)\bar{z})$ be a holomorphic motion of a smoothly bounded planar domain. Then F is trivial if and only if $a \equiv 0$ on \mathbb{B} .*

In [19], Ren-Shan Liu showed that if $f = z + t^2 \bar{z}$, then $F(\mathbb{D} \times \mathbb{D})$ is not biholomorphic equivalent to the bidisc, where \mathbb{D} denotes the unit disc. Corollary 2.7 is interesting, since every holomorphic motion of a subset of \mathbb{C} can be extended to the whole complex plane (see [25] and [26]). It is also interesting to study high-dimensional generalizations of Theorem 2.6.

3. VARIATION OF FIBRE INTEGRALS

Let \mathbb{B} be the unit ball in \mathbb{R}^m . Let $\{D_t\}_{t \in \mathbb{B}}$ be a family of smoothly bounded domain in \mathbb{R}^n . $\{D_t\}_{t \in \mathbb{B}}$ is said to be a smooth family if

$$D := \{(t, x) \in \mathbb{R}^{m+n} : x \in D_t, t \in \mathbb{B}\}$$

possesses a smooth defining function ρ such that $\rho|_{D_t}$ is a smooth defining function of D_t for every t in \mathbb{B} . Put

$$(3.1) \quad [D] := \overline{D} \cap (\mathbb{B} \times \mathbb{R}^n), \quad \delta D := \partial D \cap (\mathbb{B} \times \mathbb{R}^n).$$

Let $dx := dx^1 \wedge \cdots \wedge dx^n$ denotes the Euclidean volume form on \mathbb{R}^n . Fix a smooth function f on a neighborhood of $[D]$, the fibre integrals

$$F(t) := \int_{D_t} f(t, x) dx$$

depend smoothly on $t \in \mathbb{B}$. We shall introduce a natural way to compute the derivatives of $F(t)$ (see [24] for related results). For very fixed $j \in \{1, \dots, m\}$, let

$$V_j := \frac{\partial}{\partial t^j} - \sum v_j^\alpha \frac{\partial}{\partial x^\alpha}$$

be a smooth vector field on a neighborhood of $[D]$. We shall prove that:

Theorem 3.1. *Let $\{D_t\}_{t \in \mathbb{B}}$ be a smooth family of smoothly bounded domain in \mathbb{R}^m . Assume that $V_j(\rho)$ vanishes on δD , then we have*

$$(3.2) \quad \frac{\partial F}{\partial t^j}(t) = \int_{D_t} L_{V_j}^t(f(t, x) dx),$$

for every t in \mathbb{B} .

Proof. Without lose of generality, we may assume that $t = 0$ and $j = 1$. Since $V_1(\rho)$ vanishes on δD , the motion

$$\Phi : (-1, 1) \times D_0 \rightarrow \mathbb{R}^m$$

of D_0 associated to V_1 is compatible with $\{D_t\}$, i.e.

$$\Phi(a \times D_0) = D_{a\nu}, \quad \nu = (1, 0, \dots, 0) \in \mathbb{R}^p,$$

for every $a \in (-1, 1)$. Since for every fixed $a \in (-1, 1)$,

$$\Phi^a : x \mapsto \Phi(a, x)$$

is a C^∞ isomorphism from D_0 to $D_{a\nu}$, we have

$$(3.3) \quad \frac{\partial F}{\partial t^1}(0) = \lim_{0 \neq a \rightarrow 0} \int_{D_0} \frac{f(a\nu, \Phi^a(x)) d\Phi^a(x) - f(0, x) dx}{a}$$

Since V_1 and f are smooth up to the boundary, we have

$$(3.4) \quad \frac{\partial F}{\partial t^1}(0) = \int_{D_0} \lim_{0 \neq a \rightarrow 0} \frac{f(a\nu, \Phi^a(x)) d\Phi^a(x) - f(0, x) dx}{a}.$$

By definition of Lie derivative,

$$(3.5) \quad L_{V_1}(f(t, x) dx)(0, x) = \lim_{0 \neq a \rightarrow 0} \frac{[(\Psi^a)^*(f dx)](0, x) - f(0, x) dx}{a},$$

where

$$\Psi^a : (b\nu, \Phi^b(x)) \mapsto (b\nu + a\nu, \Phi^{b+a}(x)), \quad (b, x) \in (-1 + |a|, 1 - |a|) \times D_0.$$

Since

$$i_0^* \{[(\Psi^a)^*(f dx)](0, x) - f(a\nu, \Phi^a(x)) d\Phi^a(x)\} = 0,$$

(3.2) follows from (3.4) and (3.5). □

If $m = 2$, put

$$\frac{\partial}{\partial t} := \frac{1}{2} \left(\frac{\partial}{\partial t^1} - i \frac{\partial}{\partial t^2} \right), \quad \frac{\partial}{\partial \bar{t}} := \frac{1}{2} \left(\frac{\partial}{\partial t^1} + i \frac{\partial}{\partial t^2} \right).$$

Let

$$V = \frac{\partial}{\partial t} - \sum v^\alpha \frac{\partial}{\partial x^\alpha}$$

be a smooth vector field on a neighborhood of $[D]$. If $V(\rho)$ vanishes on δD , then both $2\text{Re}V$ and $-2\text{Im}V$ satisfy the condition of Theorem 3.1. Thus we have:

Corollary 3.2. *Assume that $V(\rho)$ vanishes on δD , we have*

$$\frac{\partial F}{\partial t}(t) = \int_{D_t} L_V^t(f(t, x) dx), \quad \frac{\partial F}{\partial \bar{t}}(t) = \int_{D_t} L_{\bar{V}}^t(f(t, x) dx),$$

for every $t \in \mathbb{B}$.

By Cartan's formula, $L_{V_j} = d\delta_{V_j} + \delta_{V_j}d$, and Theorem 3.1, we have

$$(3.6) \quad \frac{\partial F}{\partial t^j}(t) = \int_{\partial D_t} f(t, x) \delta_{V_j} dx + \int_{D_t} \frac{\partial f}{\partial t^j}(t, x) dx.$$

One may also use Theorem 3.1 to compute variation of $\int_{\partial D_t}$. Fix a smooth form

$$g = \sum g^\alpha(t, x) \widehat{dx^\alpha}$$

on a neighborhood of $[D]$, where $\widehat{dx^\alpha}$ satisfies $dx^\alpha \wedge \widehat{dx^\alpha} = dx$. The fibre integrals

$$G(t) := \int_{\partial D_t} g$$

depend smoothly on $t \in \mathbb{B}$. Theorem 3.1 implies that

Corollary 3.3. *Assume that $V_j(\rho)$ vanishes on δD , we have*

$$\frac{\partial G}{\partial t^j}(t) = \int_{\partial D_t} \delta_{V_j} dg = \int_{\partial D_t} L_{V_j}^t g,$$

for every $t \in \mathbb{B}$.

Proof. By Stokes formula and Theorem 3.1, we have

$$\frac{\partial G}{\partial t^j}(t) = \int_{\partial D_t} L_{V_j}^t d^t g,$$

where d^t is the restriction of d to D_t . Since

$$i_t^* L_{V_j} d^t g = i_t^* L_{V_j} dg,$$

we have

$$\frac{\partial G}{\partial t^j}(t) = \int_{D_t} d\delta_{V_j} dg = \int_{\partial D_t} \delta_{V_j} dg = \int_{\partial D_t} L_{V_j}^t g.$$

The proof is complete. □

4. VARIATION OF BERGMAN KERNELS

In this section we shall prove our results on the Bergman kernel stated in section 2.

4.1. Stability of Bergman kernels. We shall give an informal proof of Lemma 2.1 by using regularity properties of **full** $\bar{\partial}$ -Neumann problem (see (4.2) below). By Lemma 2.1 in [1], stability of Bergman kernels follows directly from stability of solutions u^t of a family of $\bar{\partial}$ -Neumann problems $\square^t(\cdot) = f^t$. However, in general, it is not easy to show that u^t is stable, i.e., if we want to use

$$(4.1) \quad \|\square^t(u^t - u^s)\| = \|f^t - f^s - (\square^t - \square^s)u^s\|$$

to estimate $\|u^t - u^s\|$, we have to find a natural connection between the domain of \square^t and the domain of \square^s (i.e., u^s may not be in the domain of \square^t), but then we go back to regularity properties of \square_b -equation (see [17]).

Hamilton [12] found a more natural way to study regularity properties of families of non-coercive boundary value problems (not only for $\bar{\partial}$ -Neumann problem). For reader's convenience we give a sketch description of Hamilton's idea.

Instead of considering \square^t (whose domain satisfies the $\bar{\partial}$ -Neumann condition), Hamilton considered the full Laplace operator $\widetilde{\square}^t$ (whose domain contains all forms smooth up to the boundary). Let u^t be a form smooth up to the boundary, in general, the Sobolev norm of $\widetilde{\square}^t(u^t)$ could not control the Sobolev norm of u^t . In fact, u^t has to be in the domain of \square^t (see [9]). Thus two more operators (sending forms on $\overline{D_t}$ to forms on the boundary of D_t) are used in Hamilton's paper, i.e., he considered the full $\bar{\partial}$ -Neumann problem

$$(4.2) \quad \mathfrak{S}^t(\cdot) := \left(\widetilde{\square}^t, (\bar{\partial}^t \rho) \vee, (\bar{\partial}^t \rho) \vee \bar{\partial}^t \right) (\cdot) = f^t,$$

where $(\bar{\partial}^t \rho) \vee \cdot := (\bar{\partial}^t \rho \wedge \cdot)^*$. Now the domain of \mathfrak{S}^t is $\mathbb{C}_{\bullet, \bullet}^\infty(\overline{D_t})$ for each t . Using C^∞ trivialization mapping $\mathbb{B} \times D_0 \rightarrow D$, the domain of \mathfrak{S}^t can be seen as a fixed space $\mathbb{C}_{\bullet, \bullet}^\infty(\overline{D_0})$. Thus (4.1) applies. The only thing left to do is to show that universal constant (i.e., independent of t) works in the basic estimates for \mathfrak{S}^t . It is one of main results in [12]. The interested reader is referred to that paper for further information and a clear proof.

4.2. L^2 -estimates for $\bar{\partial}a = \partial_\phi b + c$. Let X be an n -dimensional complex manifold with complete Kähler metric ω . Let ϕ be a smooth plurisubharmonic function on X . Denote by \square' (resp. \square'') the ∂_ϕ -Laplace (resp. $\bar{\partial}$ -Laplace). Let a (resp. b) be a smooth L^2 -integrable $(n, 0)$ (resp. $(n-1, 1)$) form on X such that $\bar{\partial}a$ (resp. $\partial_\phi b$) is L^2 on X (here L^2 means L^2 -integrable with respect to ω and $e^{-\phi}$). Assume that $a \perp \ker \bar{\partial}$ and b is a $\bar{\partial}$ -closed primitive form. Put $c = \bar{\partial}a - \partial_\phi b$. Since $\partial_\phi^* = - * \bar{\partial}^*$, we know that b is ∂_ϕ^* -closed. Thus b has orthogonal decomposition

$$b = b_1 + \partial_\phi^* b_2,$$

where b_1 is the \square' -harmonic part of b . We shall prove the following Lemma (due to Berndtsson [4]) for reader's convenience.

Lemma 4.1. *If ϕ and c are zero on X then $\|b\|^2 = \|a\|^2 + \|b_1\|^2$.*

Proof. Denote by G the Green operator with respect to \square'' . Since a is L^2 -minimal, we have

$$a = \bar{\partial}^* G \partial b.$$

Thus

$$\|a\|^2 = \langle G \partial b, \partial b \rangle.$$

Since ω is Kähler and $\phi \equiv 0$, we have $\square'' = \square'$. Thus $G \partial b = b_2$, which implies that $\|a\|^2 = \|\partial^* b_2\|^2 = \|b\|^2 - \|b_1\|^2$. The proof is complete. \square

We remark that by a similar argument, one may show that

$$(4.3) \quad \langle\langle b^1, b^2 \rangle\rangle = \langle\langle a^1, a^2 \rangle\rangle + \langle\langle b_1^1, b_1^2 \rangle\rangle.$$

If c is zero and ϕ is not assumed to be zero. Using Kähler identity $\square'' - \square' = [i\partial\bar{\partial}\phi, \Lambda]$, where Λ denotes the adjoint of $\omega \wedge \cdot$, one can also get an equality similar as (4.3). If c is not zero, we failed to find an equality as (4.3). However, by using Hörmander's L^2 -estimates, we get an inequality between a , b and c .

Lemma 4.2. *If $i\partial\bar{\partial}\phi > 0$ on X then $\|a\|^2 \leq \|b\|^2 - \|b_1\|^2 + \|c\|_{i\partial\bar{\partial}\phi}^2$.*

Proof. Put $u = \partial_\phi b + c$. By Hörmander's theory, it suffices to estimate

$$\langle\langle f, u \rangle\rangle = \langle\langle f, \partial_\phi(b - b_1) + c \rangle\rangle,$$

where f is an arbitrary smooth $(n, 1)$ -form with compact support. The right hand side of the above inequality is $\langle\langle \partial_\phi^* f, (b - b_1) \rangle\rangle + \langle\langle f, c \rangle\rangle$. Thus

$$|\langle\langle f, u \rangle\rangle|^2 \leq (\|\partial_\phi^* f\|^2 + \langle\langle [i\partial\bar{\partial}\phi, \Lambda]f, f \rangle\rangle) (\|b - b_1\|^2 + \|c\|_{i\partial\bar{\partial}\phi}^2).$$

Since $\square'' = \square' + [i\partial\bar{\partial}\phi, \Lambda]$, the right hand side of the above inequality is equal to

$$(\|\bar{\partial}^* f\|^2 + \|\bar{\partial} f\|^2) (\|b - b_1\|^2 + \|c\|_{i\partial\bar{\partial}\phi}^2).$$

Since ω is complete, we have

$$|\langle\langle f, u \rangle\rangle|^2 \leq (\|\bar{\partial}^* f\|^2 + \|\bar{\partial} f\|^2) (\|b - b_1\|^2 + \|c\|_{i\partial\bar{\partial}\phi}^2)$$

for every f in the domain of $\bar{\partial} \oplus \bar{\partial}^*$. Since a is the L^2 -minimal solution of $\bar{\partial}(\cdot) = u$. The proof is complete. \square

4.3. Second order variational formulas. We shall prove Theorem 2.2 in this section. In order to use L^2 -estimates in the previous section, we have to choose a suitable vector field, i.e., to prove our Key Lemma.

Proof of Key Lemma. Put $\psi = -\log -\rho$. By definition,

$$V_j^\psi = \partial/\partial t^j - \sum v_j^\alpha \partial/\partial \mu^\alpha,$$

where

$$v_j^\alpha = \sum \psi_{j\bar{\beta}} \psi^{\bar{\beta}\alpha}.$$

If $n = 1$, it is easy to check that

$$(4.4) \quad V_j^\psi := \frac{\partial}{\partial t^j} - \frac{\rho_j \rho_{\bar{\mu}} - \rho \rho_{j\bar{\mu}}}{|\rho_\mu|^2 - \rho \rho_{\mu\bar{\mu}}} \frac{\partial}{\partial \mu},$$

thus V_j^ψ satisfies our Key Lemma. If $n \geq 2$, fix $x_0 \in \partial D_0$. Choosing suitable local coordinates around x_0 , we may assume that

$$(\rho_{\alpha\bar{\beta}}(x_0)) = I_n, \quad \rho_\nu(x_0) = 0, \quad \forall \nu \geq 2,$$

where I_n is the identity matrix. Thus

$$v_j^1(x_0) = \frac{\rho_j \rho_{\bar{1}} - \rho \rho_{j\bar{1}}}{|\rho_1|^2 - \rho} (x_0) = \frac{\rho_j}{\rho_1} (x_0), \quad v_j^\alpha(x_0) = \rho_{j\bar{\alpha}}(x_0), \quad \forall \alpha \geq 2,$$

which implies that V_j^ψ is smooth up to the boundary (one may also prove this by force). Now

$$V_j^\psi(\rho)(x_0) = \rho_j(x_0) - \sum v_j^\alpha \rho_\alpha(x_0) = \rho_j(x_0) - \rho_j(x_0) = 0.$$

The proof of Key Lemma is complete. \square

Now we can use the vector fields V_j^ψ in our Key Lemma to compute variation of Bergman kernels. By (2.5) and Lemma 4.1, the last term in (2.4) is equal to the last term in (2.9). Thus Theorem 2.2 follows from the following (integration by parts) lemma:

Lemma 4.3. *If $\phi \equiv 0$ then we have*

$$(4.5) \quad \langle \langle [L_j, L_{\bar{k}}] \cdot \bar{\eta}, \cdot \bar{\zeta} \rangle \rangle = \int_{\partial D_t} b_{j\bar{k}}(\rho) \langle \cdot \bar{\eta}, \cdot \bar{\zeta} \rangle d\sigma.$$

For general ϕ with well defined geodesic curvature $c_{j\bar{k}}(\phi)$, we have

$$(4.6) \quad \langle \langle [L_j, L_{\bar{k}}] \cdot \bar{\eta}, \cdot \bar{\zeta} \rangle \rangle = \int_{\partial D_t} b_{j\bar{k}}(\rho) \langle \cdot \bar{\eta}, \cdot \bar{\zeta} \rangle d\sigma + \langle \langle c_{j\bar{k}}(\phi, V) \cdot \bar{\eta}, \cdot \bar{\zeta} \rangle \rangle,$$

where

$$c_{j\bar{k}}(\phi, V) := c_{j\bar{k}}(\phi) + \langle (\bar{\partial}^t \phi)_{V_j^\psi}, (\bar{\partial}^t \phi)_{V_{\bar{k}}^\psi} \rangle_{i\partial^t \bar{\partial}^t \phi}.$$

Proof. We shall only prove the following special case: $j = k = m = 1$, since the general case follows by a similar argument.

Put $V = V_1^\psi$. Notice that $[L_1, L_{\bar{1}}] = \bar{V}V\phi + [L_V^t, L_{\bar{V}}^t]$. By Cartan's formula, we have $(L_V - L_V^t) \cdot \bar{\eta} = \delta_{d_t V} \cdot \bar{\eta}$, where $d_t := \frac{\partial}{\partial t} \otimes dt + \frac{\partial}{\partial \bar{t}} \otimes d\bar{t}$. Thus

$$(L_{\bar{V}}^t L_V^t \cdot \bar{\eta})_{(n,0)} = (L_{\bar{V}} L_V \cdot \bar{\eta})_{(n,0)},$$

where $(\cdot)_{(n,0)}$ means the $(n,0)$ -component of $i_t^*(\cdot)$. Since $L_{\bar{V}} \cdot \bar{\eta} = \cdot \bar{\eta}_{\bar{t}} = L_{\bar{V}}^t \cdot \bar{\eta}$, we have

$$([L_V^t, L_{\bar{V}}^t] \cdot \bar{\eta})_{(n,0)} = ([L_V, L_{\bar{V}}] \cdot \bar{\eta})_{(n,0)}.$$

Since $[L_V, L_{\bar{V}}] = L_{[V, \bar{V}]}$, we have

$$([L_V^t, \mathcal{L}_{\bar{V}}^t] \cdot \bar{\eta})_{(n,0)} = \partial^t \delta_{[V, \bar{V}]} \cdot \bar{\eta}.$$

Put $V = \frac{\partial}{\partial t} - \sum v^\alpha \frac{\partial}{\partial \mu^\alpha}$, we have

$$\begin{aligned} \bar{V}V\phi - \langle (\bar{\partial}^t \phi)_V, (\bar{\partial}^t \phi)_V \rangle_{i\partial^t \bar{\partial}^t \phi} &= c(\phi) - \sum \left(v_t^\alpha \phi_\alpha - \bar{v}^\beta v_{\bar{\beta}}^\alpha \phi_\alpha \right) \\ &= c(\phi) - \delta_{[V, \bar{V}]} \partial^t \phi, \end{aligned}$$

where $c(\phi)$ is short for $c_{1\bar{1}}(\phi)$. Thus

$$([L_1, L_{\bar{1}}] \cdot \bar{\eta})_{(n,0)} = c(\phi, V) \cdot \bar{\eta} + \partial_\phi^t \delta_{[V, \bar{V}]} \cdot \bar{\eta},$$

where $c(\phi, V)$ is short for $c_{1\bar{1}}(\phi, V)$. It suffices to show that

$$(4.7) \quad \langle \langle \partial_\phi^t \delta_{[V, \bar{V}]} \cdot \bar{\eta}, \cdot \bar{\zeta} \rangle \rangle = \int_{\partial D_t} b(\rho) \langle \cdot \bar{\eta}, \cdot \bar{\zeta} \rangle d\sigma,$$

where $b(\rho)$ is short for $b_{1\bar{1}}(\rho)$. Notice that the left hand side of the above equality is equal to

$$i^{n^2} \int_{\partial D_t} \{ \delta_{[V, \bar{V}]} \cdot \bar{\eta}, \cdot \bar{\zeta} \} = \int_{\partial D_t} \frac{\sum (V \bar{v}^\alpha) \rho_{\bar{\alpha}}}{|\partial \rho|} \langle \cdot \bar{\eta}, \cdot \bar{\zeta} \rangle d\sigma.$$

Since

$$\langle V, V \rangle_{i\partial \bar{\partial} \rho} - \sum (V \bar{v}^\alpha) \rho_{\bar{\alpha}} = V \bar{V} \rho$$

and $V \bar{V} \rho = 0$ on the boundary of D_t , we get (4.7). The proof is complete. \square

We remark that the only property of the vector field V_j^ψ used in the proof of the above lemma is $V_j^\psi \rho = 0$ on the boundary. Thus the above lemma is true for every $V_j \in \mathcal{V}_j$.

Notice that if every fiber is one-dimensional then every vector field in \mathcal{V}_j can be used to compute the variation. What's more,

$$\frac{\langle V_j, V_k \rangle_{i\partial\bar{\partial}\rho}}{|\partial\rho|} \equiv \frac{\rho_{j\bar{k}}|\rho_\mu|^2 - \rho_{j\bar{\mu}}\rho_{\bar{k}}\rho_\mu - \rho_{\bar{k}\mu}\rho_j\rho_{\bar{\mu}} + \rho_j\rho_{\bar{k}}\rho_{\mu\bar{\mu}}}{|\rho_\mu|^3}$$

does not depend on $V_j \in \mathcal{V}_j$, $V_k \in \mathcal{V}_k$. Thus we have

Theorem 4.4. *Let $\{D_t\}_{t \in \mathbb{B}}$ be a smooth family of smoothly bounded planar domains. If $\phi \equiv 0$ on D then we have*

$$K_{j\bar{k}}(\zeta, \bar{\eta}) = \int_{\partial D_t} b_{j\bar{k}}(\rho) \langle \cdot, \bar{\eta} \rangle \langle \cdot, \bar{\zeta} \rangle d\sigma + \langle \langle \cdot, \bar{\eta}_{\bar{k}} \rangle, \bar{\zeta}_{\bar{j}} \rangle \rangle + \langle \langle \cdot, \bar{\eta}_{\bar{j}} \rangle, \bar{\zeta}_{\bar{k}} \rangle \rangle$$

where $j\bar{\eta}$ is the harmonic part of $\delta_{\bar{\partial}^t V_j} \cdot \bar{\eta}$.

We shall use the above theorem to study triviality of holomorphic motions.

4.4. Bergman kernel and curvature property. We shall prove Corollary 2.3, Theorem 2.4 and Theorem 2.5 in this section.

Let $\pi : D \rightarrow \mathbb{B}$ be a proper holomorphic submersion. Let's recall the definition of Nakano positivity for holomorphic vector bundle $\{\mathcal{H}_t\}$ associated to 0-th direct image $\pi_*(K_{D/\mathbb{B}})$ (we assume that $\dim_{\mathbb{C}} \mathcal{H}_t$ is a constant r). Thus \mathcal{H}_t can be seen as the Bergman space of the fibre at t . By definition, $\{\mathcal{H}_t\}$ is said to be Nakano semi-positive if for every $u^1, \dots, u^m \in \mathcal{H}_t$,

$$(4.8) \quad \sum \langle \langle \Theta_{j\bar{k}} u^j, u^k \rangle \rangle \geq 0,$$

where $\Theta_{j\bar{k}}$ is the curvature of the Chern connection on $\{\mathcal{H}_t\}$. We may choose $\bar{\eta}_p$, $p = 1, \dots, r$, such that $\mathcal{H}_t = \text{Span}\{\bar{\eta}_p\}$. Thus every u^j can be written as $u^j = \sum c^{jp} \bar{\eta}_p$. Hence (4.8) is equivalent to

$$(4.9) \quad \sum c^{jp} \overline{c^{kq}} \langle \langle \Theta_{j\bar{k}} \cdot \bar{\eta}_p, \bar{\eta}_q \rangle \rangle \geq 0.$$

Denote by D_j the contraction of $\partial/\partial t^j$ with $(1, 0)$ -component of the Chern connection on $\{\mathcal{H}_t\}$. By definition of the Chern connection, $D_j \cdot \bar{\eta}_p$ is the Bergman projection of $L_j \cdot \bar{\eta}_p$. By (2.1), $L_j \cdot \bar{\eta}_p \perp \mathcal{H}_t$. Thus $D_j \cdot \bar{\eta}_p = 0$. Since $\Theta_{j\bar{k}} = [D_j, \partial/\partial \bar{t}^k]$, we have

$$\langle \langle \Theta_{j\bar{k}} \cdot \bar{\eta}_p, \bar{\eta}_q \rangle \rangle = K_{j\bar{k}p\bar{q}}.$$

Thus we get the following lemma:

Lemma 4.5. *Nakano semi-positivity of $\{\mathcal{H}_t\}$ is equivalent to (2.10).*

In case $\dim_{\mathbb{C}} \mathcal{H}_t = \infty$, $\{\mathcal{H}_t\}$ is said to be Nakano semi-positive if (2.10) is true for every positive integer r . Thus Corollary 2.3 is a direct corollary of Lemma 4.3 and Theorem 2.2.

Proof of Theorem 2.4. Since D is pseudoconvex and ϕ is plurisubharmonic, by Lemma 4.3, we have

$$\sum c^{jp} \overline{c^{kq}} \langle \langle [L_j, L_{\bar{k}}] \cdot \bar{\eta}_p, \bar{\eta}_q \rangle \rangle \geq \|c\|_{i\partial\bar{\partial}^t \phi}^2,$$

where

$$c := \sum c^{jp} (\bar{\partial}^t \phi)_{V_j^\psi} \wedge \bar{\eta}_p.$$

The last term in (2.4) can be written as $\|b\|^2 - \|a\|^2$, where

$$b = \sum c^{jp}_j \bar{\eta}_p^{n-1,1}, \quad a = - \sum c^{jp}_j \bar{\eta}_p^{n,0}.$$

Thus it suffices to show that $\|b\|^2 - \|a\|^2 + \|c\|_{i\partial^t \bar{\partial}^t \phi}^2 \geq 0$. By (2.5), $\bar{\partial}^t a = \partial_\phi^t b + c$. Thus Theorem 2.4 follows from Lemma 4.2. \square

Proof of Theorem 2.5. We may assume $m = 1$. Put

$$P_f(z) = \int_{D_t} K^t(z, \bar{w}) \overline{f(t, w)}.$$

We claim that $D_t P_f := e^\phi \partial / \partial t (P_f e^{-\phi})$ is perpendicular to the Bergman space: It suffices to show that $\langle \langle h, D_t(P_f) \rangle \rangle = 0$ for every function h holomorphic on a neighborhood of the closure of D_t . Thus h can be seen as a holomorphic function on nearby fibres and

$$0 = \partial / \partial \bar{t} \int_{D_t} h f = \partial / \partial \bar{t} \int_{D_t} h \bar{P}_f e^{-\phi} = \langle \langle h, D_t P_f \rangle \rangle.$$

Our claim is proved.

Since $K_f(t) = \|P_f\|^2$, we have

$$K_{f,t\bar{t}} = \|P_{f,\bar{t}}\|^2 + \langle \langle \phi_{t\bar{t}} P_f, P_f \rangle \rangle - \|D_t P_f\|^2.$$

Notice that $\bar{\partial}^t(D_t P_f) = -P_f \bar{\partial}^t(\phi_t)$. By Lemma 4.2,

$$\|D_t P_f\|^2 \leq \langle \langle \bar{\partial}^t \phi_t |_{i\partial^t \bar{\partial}^t \phi}^2 P_f, P_f \rangle \rangle.$$

Thus

$$K_{f,t\bar{t}} \geq \|P_{f,\bar{t}}\|^2 + \langle \langle c(\phi) P_f, P_f \rangle \rangle,$$

which implies that

$$(\log K_f)_{t\bar{t}} \geq \frac{\langle \langle c(\phi) P_f, P_f \rangle \rangle}{\|P_f\|^2} \geq 0.$$

The proof is complete. \square

Now we shall use Berndtsson's approximation technique (see section 3 in [1]) to prove the remark behind Theorem 2.5.

Proof of the remark behind Theorem 2.5. Notice that K_f satisfies the following extremal property:

$$K_f(t) = \sup_{h \in \mathcal{H}_t} \left\{ \left| \int_{D_t} h f \right|^2 / \|h\|^2 \right\}.$$

Since the Bergman kernel associated to ϕ and D is a decreasing limit of the Bergman kernel associated to smooth weight and smooth strictly pseudoconvex domain. We know that $\log K_f$ is plurisubharmonic on \mathbb{B} for general ϕ and D . \square

5. APPLICATIONS TO HOLOMORPHIC MOTIONS

We shall prove Theorem 2.6 in this section. Let D_0 be a smoothly bounded planar domain. Let F be a holomorphic motion of D_0 . If F is smooth up to the boundary, then the vector field V_j^F defined by $V_j^F := F_* \left(\frac{\partial}{\partial t^j} \right)$ is tangent to the boundary (i.e., $V_j^F \in \mathcal{V}_j$). Thus Theorem 4.4 applies. Since the graph D of F is Levi-flat, we have

$$\sum c^{jp} \overline{c^{kq}} \int_{\partial D_t} b_{j\bar{k}}(\rho) \langle \cdot, \bar{\eta}_p \rangle \langle \cdot, \bar{\eta}_q \rangle d\sigma = 0,$$

which implies that

$$(5.1) \quad \sum c^{jp} \overline{c^{kq}} K_{j\bar{k}p\bar{q}} = || \sum c^{jp} \bar{\eta}_p ||^2.$$

Proof of Theorem 2.6. (i) \Rightarrow (ii): If F is trivial then there exists a bi-holomorphic mapping G with the same fibres. Thus V_j^G are holomorphic. Hence ${}_j\bar{\eta}_p \equiv 0$. Then (5.1) implies (ii).

(ii) \Rightarrow (iii): If (ii) is true then ${}_j\bar{\eta} \equiv 0$ by (5.1). Thus $\delta_{\partial^t V_j^F} \cdot \bar{\eta} \perp \ker \partial^t$. Since $d\bar{\zeta} \in \ker \partial^t$, we have

$$(5.2) \quad \int_{D_t} f_{j\bar{\zeta}} K^t(\zeta, \bar{\eta}) id\zeta \wedge d\bar{\zeta} = 0.$$

Notice that

$$\overline{z_\zeta} = \frac{f_z}{|f_z|^2 - |f_{\bar{z}}|^2}, \quad z_{\bar{\zeta}} = \frac{-f_{\bar{z}}}{|f_z|^2 - |f_{\bar{z}}|^2},$$

we have

$$(5.3) \quad f_{j\bar{\zeta}} = f_{jz} z_{\bar{\zeta}} + f_{j\bar{z}} \overline{z_\zeta} = \frac{(f_z)^2 J_j}{|f_z|^2 (1 - |J|^2)}.$$

Thus (iii) is true.

(iii) \Rightarrow (i): It suffices to find holomorphic vector fields V_j , $j = 1, \dots, m$, on D such that $V_j = V_j^F$ on ∂D and $[V_j, V_k] = 0$ on D . By (5.2), there exists $g^{t,j}$ such that

$$(5.4) \quad \bar{\partial}^t f_j = (\partial^t)^* (g^{t,j} id\zeta \wedge d\bar{\zeta}) = -i \bar{\partial}^t g^{t,j}.$$

Take g^j such that $g^j|_{D_t} = g^{t,j}$. We claim that

$$V_j = \partial/\partial t^j + (f_j + ig^j) \partial/\partial \zeta, \quad j = 1, \dots, m,$$

fit our needs. Since $g^{t,j} id\zeta \wedge d\bar{\zeta} \in \text{Dom}(\partial^t)^*$, we have $g^j = 0$ on ∂D (i.e., $V_j = V_j^F$ on ∂D). Thus it suffices to prove that V_j are holomorphic and integrable.

By (5.4), $f_{j\bar{k}} + ig_k^j$ are holomorphic on each fibre. To prove $f_{j\bar{k}} + ig_k^j = 0$ on each fibre, it suffices to show that $f_{j\bar{k}} + ig_k^j = 0$ on the boundary of each fibre.

Since $g^j = 0$ on ∂D , we have

$$(5.5) \quad V_k^F g^j = g_k^j + f_k g_\zeta^j = 0, \quad \text{on } \partial D,$$

and

$$(5.6) \quad \overline{V_k^F} g^j = g_{\bar{k}}^j + \overline{f_k} g_{\bar{\zeta}}^j = 0, \quad \text{on } \partial D.$$

By definition of V_j^F , we have $[V_j^F, V_k^F] = 0$ and $[V_j^F, \overline{V_k^F}] = 0$. Thus

$$(5.7) \quad f_j f_{k\zeta} - f_k f_{j\zeta} = 0, \quad f_{j\bar{k}} + \overline{f_k} f_{j\bar{\zeta}} = 0, \quad \text{on } D.$$

By (5.4), (5.6) and (5.7), we have

$$0 = \overline{f_k} f_{j\bar{\zeta}} + i \overline{f_k} g_{\bar{\zeta}}^j = -f_{j\bar{k}} - i g_{\bar{k}}^j, \quad \text{on } \partial D.$$

Thus $f_{j\bar{k}} + i g_{\bar{k}}^j \equiv 0$ on D . By (5.4), V_j are holomorphic on D .

Now we need to show that $[V_j, V_k] = 0$ on D . Since $[V_j, V_k]$ are holomorphic, it suffices to show that $[V_j, V_k] = 0$ on ∂D . Since $g^j = 0$ on ∂D , we have

$$[V_j, V_k] = f_j f_{k\zeta} - f_k f_{j\zeta} + i(f_j g_{\zeta}^k + g_j^k) - i(f_k g_{\zeta}^j + g_k^j), \quad \text{on } \partial D.$$

By (5.5) and (5.7), $[V_j, V_k] = 0$ on ∂D . Thus V_j are integrable and our claim is proved. The proof is complete. \square

Proof of Corollary 2.7. By definition of F , (iii) is equivalent to $a_j(t) \equiv 0$. Since $a(0) = 0$, (iii) is equivalent to $a \equiv 0$. Thus Corollary 2.7 follows from Theorem 2.6. \square

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